

# Quantum Bundles and Noncommutative Complex Structures

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## Abstract

We introduce a novel first-order differential calculus for the quantum unitary group  $C_q[U_N]$ , and a generalisation of the Woronowicz construction of higher forms which is applicable to it. The exterior algebra produced is shown to have classical dimension. Building on this we describe the quantum projective spaces (endowed with the Heckenberger–Kolb calculus) as the base space of a quantum homogeneous bundle with total space  $C_q[U_N]$ . The induced higher forms for each quantum projective space are then framed as associated bundles using a new general result. Next a notion of noncommutative complex structure is given, along with a general method for constructing such structures from quantum homogeneous bundles. The method is applied to the quantum projective spaces yielding a direct deformation of their classical complex structure.

## 1 Introduction

The interaction of the theory of quantum groups with Connes’ formulation of noncommutative geometry is a very important, exciting, and active area of contemporary mathematics. Both theories are based on a philosophy of viewing noncommutative algebras as generalisations of function algebras. Thus, it is natural to expect that they can be understood in some unified way. However, major difficulties arise when one tries to find such a common viewpoint.

The central notion in Connes’ approach to noncommutative geometry is that of a *spectral triple*: This is a type of noncommutative generalisation of the Dirac operator on a Riemannian spin manifold [7, 15]. The main role of spectral triples is to provide a means to calculate the index pairing between the K-theory and K-homology groups of  $C^*$ -algebras, using techniques that generalise the methods of the Atiyah–Singer index theorem.

However, it is notoriously difficult to construct generalised Dirac operators on quantum groups, or quantum group homogeneous spaces, that respect the spectral triple axioms. Indeed, for some time it was even suspected that it was impossible to do so. The first result to suggest otherwise was obtained by Chakraborty and Pal [5], who constructed a spectral triple on  $C_q[SU_2]$  following the isospectral deformation approach suggested by Connes and Landi [9]. A large number of examples would follow, including spectral

triples on general  $\mathbf{C}_q[SU_N]$ , on the odd and even dimensional quantum spheres, and on all the quantum projective spaces [6, 10, 11, 12, 13]. One of most notable results was that of Neshveyev and Tuset [39], who used a Drinfeld twist to construct spectral triples for all the Drinfeld–Jimbo quantum groups and their homogeneous spaces. However, since the construction of a Drinfeld twist is not particularly explicit, certain properties of these Dirac operators are not immediate. Moreover, it is not clear that the spectral triples adequately take the deformation of the quantum groups into account.

Now before anyone had ever tried to construct Dirac operators on quantum groups, there already existed a purely algebraic approach to studying the geometry of noncommutative Hopf algebras. This approach has its origin in the seminal work of S. L. Woronowicz [49]. The basic structure needed here is that of a *covariant differential calculus*, which generalises the Kähler differential forms of a variety. Noncommutative versions of the objects from classical differential geometry are then built up piece by piece in terms of this structure.

The main problem with Woronowicz’s approach is that for a general quantum group, or quantum group homogeneous space, reasonable covariant calculi can often fail to exist, leaving doubt about the best way to proceed. However, there exists a major family of quantum group homogeneous spaces for which this problem does not arise: the quantum flag manifolds [30, 47, 48]. In [18], it was shown that there exist exactly two finite-dimensional irreducible covariant calculi for each  $\mathbf{C}_q[G/P]$ . One of these is a  $q$ -deformation of the holomorphic forms of  $G/P$ , while the other is a  $q$ -deformation of the anti-holomorphic forms. Their direct sum is usually called the *Heckenberger–Kolb calculus*. The unambiguous nature of this result shows that Woronowicz’s theory of covariant differential calculi is intimately suited to the study of the geometry of quantum flag manifolds. Moreover, the Heckenberger–Kolb calculus canonically extends to a total differential calculus with a complex structure directly generalising the complex structure of the undeformed flag manifolds [19].

Classically, each flag manifold has a canonical metric coming from the restriction of the Fubini–Study metric for the projective spaces. This metric is Kähler, and so, it has an associated Dirac operator which is a twist of the Dirac–Dolbeault operator  $\bar{\partial} + \bar{\partial}^*$ . Thus, an obvious starting point for the construction of Dirac operators for the quantum flag manifolds would be try and find a metric and Hodge operator for the Heckenberger–Kolb total calculus. For a Hopf algebra  $G$ , with a left-covariant calculus  $\Omega^1(G)$ , there is a natural way to do this [17]. The method uses the Woronowicz Theorem [49], which describes each such left-covariant calculus as a free module  $G \otimes V$ , where  $V$  is a vector space generalising the cotangent space. However, calculi for homogeneous spaces are not in general free modules (classically this corresponds to the fact that they are not in general parallelisable). The calculi are usually given a presentation in terms of generators and relations which can be difficult to use as a base for further progress. Indeed, subsequent work in the area by Krähmer [29], and by Dąbrowski and D’Andrea [8, 14], departed from the direct differential calculus approach and used quite different methods. The Dirac operators produced, however, had a number of shortcomings, casting doubt on the suitability of their constructions.

Another approach to the problem is the theory of *quantum framed manifolds* introduced by Majid [37, 35]. Here one looks for a description of the calculus as a quantum associated bundle. The approach seeks to generalise the fact that while every cotangent bundle is not parrellisable, it is always expressable as an associated bundle to a principal bundle. Associated bundles generalise to the noncommutative setting as coinvariant submodules of free modules  $G \otimes V$ , where again  $G$  is a Hopf algebra, and  $V$  a vector space generalising the cotangent space. With such a description of a calculus it is possible to begin constructing Hodge  $*$ -operators using a generalisation of the approach in [17].

In [40] a quantum framed manifold description of the quantum projective spaces, endowed with the first order Heckenberger–Kolb calculus, was given. This generalised the quantum framed manifold description of the quantum projective line (or Podleś sphere), endowed with the Podleś calculus, given in [37]. Moreover, it gave for the first time a description of the holomorphic and anti-holomorphic calculi of the quantum projective spaces as associated bundles. The goal of this paper is two fold: First it aims to extend the framed description of the one forms, to a framed description of the higher forms. Second, it seeks to build on this to produce a systematic frame bundle approach in which to understand the complex structure of the total calculi presented in [19].

What emerges naturally from this work is a noncommutative generalisation of the Kähler geometry of the quantum projective spaces. This will be thoroughly investigated in a sequel [41] in terms of a general theory of noncommutative Kähler structures. Pending this, we give a full presentation of the noncommutative Kähler geometry of the special case of the quantum projective line  $\mathbf{C}_q[\mathbf{CP}^1]$ .

The paper is organised as follows: Section 2 is preliminary. It introduces basic material about bicovariant differential calculi over Hopf algebras, quantum homogeneous bundles, and quantum framed manifolds.

In Section 3 we recall the basic details of the quantum group  $\mathbf{C}_q[U_N]$ , and present the coinvariant subalgebra  $\mathbf{C}_q[\mathbf{CP}^{N-1}] = \mathbf{C}_q[U_N]^{\mathbf{C}_q[U_1] \otimes \mathbf{C}_q[U_{N-1}]}$ .

We construct a novel classical dimension calculus for  $\mathbf{C}_q[U_N]$  in Section 4, and describe the calculus that it induces by restriction on  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$ .

In Section 5 we extend a result in [37], to give a framing for any tensor power of a calculus on the base of a quantum homogeneous bundle (subject to a mild condition that we call right-absorbing). The framing (and the required condition) are well explored through examples.

In Section 6 we generalise the Woronowicz exterior algebra construction so as to apply it to our new calculus on  $\mathbf{C}_q[U_N]$ . The resulting graded algebra is shown to have classical dimension. The induced higher forms for each quantum projective space are then framed as associated bundles using a new general result. Moreover, using an adaptation of Woronowicz’s method, we introduce an exterior derivative for the exterior algebra of  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$ .

The heart of the paper is found in Section 7. Here we introduce our notion of almost-complex structure, and integrability. We then give a succinct set of easily verifiable

criteria for a general quantum homogeneous base calculus to have such a structure. In the case that it does, a description of each module of  $(p, q)$ -forms as an associated bundle is produced. Applying this result to the quantum projective spaces yields a direct deformation of the complex structure of the classical projective spaces.

In the final section, Section 8, we fully explore the Kähler geometry of the calculus on the quantum projective line  $\mathbf{C}_q[\mathbf{CP}^1]$ . This yields a noncommutative generalisation of the Kähler identities for  $\mathbf{CP}^1$ .

I would like to thank Edwin Beggs and Paul Smith for generously sharing with me a copy of their comprehensive preprint [1] on noncommutative complex differential geometry. Moreover, I would like to acknowledge that it is from here that I take my definition of an integrable almost-complex structure.

## 2 Preliminaries

In this section we fix notation and recall the definitions, constructions, and results that will be used later on. References are provided where proofs or basic details are omitted. Let  $A$  be an algebra. (In what follows all algebras are assumed to be unital.) A *first-order differential calculus* over  $A$  is a pair  $(\Omega^1, d)$ , where  $\Omega^1$  is an  $A$ - $A$ -bimodule and  $d : A \rightarrow \Omega^1$  is a linear map for which holds the *Leibniz rule*

$$d(ab) = a(db) + (da)b, \quad (a, b \in A),$$

and for which  $\Omega^1 = \text{span}_{\mathbf{C}}\{adb \mid a, b \in A\}$ . We call an element of  $\Omega^1(A)$  a *1-form*. The *universal first-order differential calculus* over  $A$  is the pair  $(\Omega_u^1(A), d_u)$ , where  $\Omega_u^1(A)$  is the kernel of the product map  $m : A \otimes A \rightarrow A$  endowed with the obvious bimodule structure, and  $d_u$  is defined by

$$d_u : A \rightarrow \Omega_u^1(A), \quad a \mapsto 1 \otimes a - a \otimes 1. \quad (1)$$

It is not difficult to show that every calculus over  $A$  is of the form  $(\Omega_u^1(A)/N, \text{proj} \circ d_u)$ , where  $N$  is a sub-bimodule of  $\Omega_u^1(A)$ , and  $\text{proj} : \Omega_u^1(A) \rightarrow \Omega_u^1(A)/N$  is the canonical projection.

Let  $H$  be a Hopf algebra with comultiplication  $\Delta_H$ , counit  $\varepsilon_H$ , antipode  $S_H$ , unit  $1_H$ , and multiplication  $m_H$  (where no confusion arises we will usually omit explicit reference to  $H$  when denoting these operators). A differential calculus  $\Omega^1(A)$  over a left  $H$ -comodule  $A$  is said to be *left-covariant* if there exists a left-coaction  $\Delta_L : \Omega^1(A) \rightarrow H \otimes \Omega^1(A)$  such that

$$\Delta_L(adb) = \Delta_L(a)(\text{id} \otimes d)\Delta_L(b), \quad (a, b \in A).$$

A calculus over a right  $H$ -comodule is said to be *right-covariant* if there exists an analogous right-coaction  $\Delta_R$ . A calculus over a  $H$ -bicomodule that is both left and

right-covariant is said to be *bicovariant* if  $(\text{id} \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes \text{id}) \circ \Delta_R$ . The left-covariant differential calculi over  $H$  were classified in [49] as follows: Consider the linear isomorphism

$$s : H \otimes H \rightarrow H \otimes H, \quad a \otimes b \mapsto aS(b_{(1)}) \otimes b_{(2)},$$

with inverse

$$s^{-1} : H \otimes H \rightarrow H \otimes H, \quad a \otimes b \mapsto ab_{(1)} \otimes b_{(2)}.$$

The restriction of  $s^{-1}$  to the universal calculus  $\Omega_u^1(H)$  is a linear isomorphism

$$s^{-1} : \Omega_u^1(H) \rightarrow H \otimes H^+, \quad (2)$$

where  $H^+ = \ker(\varepsilon)$  denotes the augmentation ideal of  $H$ . Now for any right ideal  $I_H$  of  $H^+$ , it can be shown that  $s(H \otimes I_H)$  is a sub-bimodule of  $\Omega_u^1(H)$  for which the corresponding calculus  $\Omega^1(H)$  is left-covariant. Moreover, it can be shown that every left-covariant calculus arises in this way. This correspondence is bijective, meaning that the left-covariant calculi over  $H$  are classified by the right ideals of  $H^+$ . If we denote  $\Lambda_H^1 = H^+/I_H$ , then it is clear that  $s$  descends to an isomorphism between  $H \otimes \Lambda_H^1$  and  $\Omega^1(H)$ . In what follows we will drop any explicit reference to  $s$  and tacitly identify these two spaces. Building upon the classification of left-covariant calculi, it can be shown that bicovariant calculi are in bijective correspondence with the  $\text{Ad}_R$ -stable right ideals of  $H^+$ , that is, right ideals  $I_H$  such that  $\text{Ad}_R(I_H) \subseteq I_H \otimes H$ , where as usual  $\text{Ad}_R(h) = h_{(2)} \otimes S(h_{(1)})h_{(3)}$ , for  $h \in H$ .

For a right  $H$ -comodule  $V$  with coaction  $\Delta_R$ , we say that an element  $v \in V$  is *coinvariant* if  $\Delta_R(v) = v \otimes 1$ , we denote the subspace of all coinvariant elements by  $V^H$ , and call it the *coinvariant subspace* of the coaction. (We define coinvariant subspace of a left-coaction analogously.) For Hopf algebras  $G, H$ , a *homogeneous* right  $H$ -coaction on  $G$  is a coaction of the form  $(\text{id} \otimes \pi) \circ \Delta$ , where  $\pi : G \rightarrow H$  is a surjective Hopf algebra map. We call the coinvariant subalgebra of such a coaction a *quantum homogeneous space*.

An *associated bundle* to a quantum homogeneous space  $\pi : G \rightarrow H$  is a coinvariant subalgebra of the form  $\mathcal{E} = (G \otimes V)^H$ , where  $V$  is a  $H$ -comodule and  $G \otimes V$  is equipped with the tensor product coaction. We say that an *algebra*  $M$ , endowed with a differential calculus  $\Omega^1(M)$ , is a *quantum homogeneous framed manifold* if it is the base of a quantum homogeneous bundle  $\pi : G \rightarrow H$  such that  $\Omega^1(G)$  induces  $\Omega^1(M)$  by restriction, and  $\Omega^1(M)$  is isomorphic to an associated bundle of  $\pi : G \rightarrow H$ . We call such an isomorphism a *framing*. For a right  $H$ -comodule  $V$ , a *soldering form* is a map  $\theta : V \rightarrow P\Omega^1(M)$  for which the induced left  $M$ -module map

$$s_\theta : (P \otimes V)^H \rightarrow P\Omega^1(M), \quad p \otimes v \mapsto p\theta(v),$$

is a framing. In general it is not clear how to find a framing, or even if one exists. However, in the case of a quantum homogeneous space we have the following theorem:

**Theorem 2.1** [37] *For any quantum homogeneous bundle  $\pi : G \rightarrow H$  with base space  $M$ , for which the vector space  $V_M = (G^+ \cap M)/(I_G \cap M)$  has a well-defined right  $H$ -comodule structure given by*

$$\Delta_M(\bar{v}) = \bar{v}_{(2)} \otimes S(\pi(v_{(1)})), \quad (v \in G^+ \cap M) \quad (3)$$

*Then a soldering form is given by*

$$\theta(\bar{v}) = S(v_{(1)})dv_{(2)}. \quad (4)$$

In what follows, we will usually denote  $M^+ = G^+ \cap M$  and  $I_M = I \cap M$ . Moreover, we define the *dimension* of  $\Omega^1(M)$  to be the dimension of  $V_M$ .

We now come to noncommutative higher differential forms: For  $G$  an additive group, a  *$G$ -graded algebra* is an algebra of the form  $A = \bigoplus_{g \in G} A^g$ , where each  $A^g$  is a linear subspace of  $A$ , and  $A^g A^f \subseteq \Omega^{f+g}$ , for all  $g, f \in G$ . If  $a \in A^g$ , then we say that  $a$  is of *degree  $g$* . A *homogenous mapping of degree  $f$*  on  $A$  to itself is a linear mapping  $h : A \rightarrow A$  such that if  $a \in A^g$ , then  $h(a) \in A^{g+f}$ . For  $G = \mathbf{N}_0$ , a *graded derivation*  $d$  on  $A$  is a homogenous mapping of degree 1 such that

$$d(ab) = d(a)b + (-1)^n adb,$$

for all  $a \in A^n$ , and  $b \in A$ . A pair  $(\Omega, d)$  is a *differential algebra* if  $\Omega$  is an  $\mathbf{N}_0$ -graded algebra and  $d$  is a graded derivation on  $\Omega$  such that  $d^2 = 0$ . The operator  $d$  is called the *differential* of the algebra.

**Definition 2.2.** A *differential calculus* over an algebra  $A$  is a differential algebra  $(\Omega, d)$ , such that  $\Omega^0 = A$ , and

$$\Omega^n = d(\Omega^{n-1}) \oplus Ad(\Omega^{n-1}), \quad n \geq 1. \quad (5)$$

If  $(\Omega(A), d)$  is a differential calculus over a  $*$ -algebra  $A$  such that the involution of  $A$  extends to an involutive conjugate-linear map  $*$  on  $\Omega$ , for which  $d(\omega^*) = (d\omega)^*$ , for all  $\omega \in \Omega$ . If it holds that

$$(\omega_p \omega_q)^* = (-1)^{pq} \omega_q^* \omega_p^*, \quad \text{for all } \omega_p \in \Omega^p, \omega_q \in \Omega^q,$$

then we say that  $(\Omega, d)$  is a *differential  $*$ -calculus*.

Classically the higher forms are constructed from the one-forms using the flip operator. However, for modules over noncommutative rings the flip operator is no longer guaranteed to be well-defined as a module map. Thus a more general procedure is required. One such procedure is the *Woronowicz exterior algebra construction*: For a Hopf algebra  $G$ , let  $\mathcal{M}_G$  be its category of right  $G$ -modules, endowed with its usual monoidal structure. (See [24, 28, 36] for details on monoidal categories.) For an object  $V \in \mathcal{M}_G$ ,

a *braiding* (or *Yang–Baxter operator*) for  $V$  is an isomorphism  $\Psi : X \otimes X \rightarrow X \otimes X$  such that  $\Psi$  satisfies the *braid relation*:

$$(\Psi \otimes \text{id}) \circ (\text{id} \circ \Psi) \circ (\Psi \otimes \text{id}) = (\text{id} \circ \Psi) \circ (\Psi \otimes \text{id}) \circ (\Psi \otimes \text{id}). \quad (6)$$

Let  $S_k$  be the group of permutations on  $k$ -objects considered as on acting on  $\{1, 2, \dots, k\}$ . Moreover, for  $a < k$ , we denote by  $t_a \in S_k$  the adjacent transposition that interchanges  $a$  and  $a + 1$ . Let us associate to  $t_a$  the morphism

$$\Psi_k = \text{id}^{\otimes(a-1)} \otimes \Psi \otimes \text{id}^{\otimes(k-a)}.$$

Recall now that every element  $\pi \in S_k$  can be expressed as a product of adjacent transpositions  $t_{a_1} \cdots t_{a_{\ell(\pi)}}$ , where  $\ell(\pi)$  is the length of the permutation  $\pi$ . This gives us a morphism  $\Pi_\pi = \Psi_{a_1} \cdots \Psi_{a_{\ell(\pi)}}$  for each  $\pi$ , which the braid relation tells us is independent of the choice of expression of  $\pi$ . From here can define the *braided  $k$ -antisymmetriser* to be the bimodule map

$$A_{\Psi,k} : \Lambda_G^{\otimes k} \rightarrow \Lambda_G^{\otimes k}, \quad \omega_1 \otimes \cdots \otimes \omega_k \mapsto \sum_{p \in S_k} \text{sgn}(p) \Pi_p \omega_1 \otimes \cdots \otimes \omega_k,$$

where  $\text{sgn}(p)$  is the sign of the permutation  $p$ . Since the kernel of  $A_{\Psi,k}$  is clearly a right submodule of  $V^{\otimes k}$ , the quotient  $V^k = V^{\otimes k} / \ker(A_{\Psi,k})$  is well-defined as a right module. Moreover, using elementary permutation theory arguments, it can be shown that  $\bigoplus_{k=1}^{\infty} A_{\Psi,k}$  is a right submodule of  $\bigoplus_{k=1}^{\infty} V^{\otimes k}$ . Thus,

$$\bigoplus_{k=1}^{\infty} V^k = \bigoplus_{k=1}^{\infty} V^{\otimes k} / \ker(A_{\Psi,k})$$

is well-defined as an algebra. We call it the *exterior algebra of  $V$*  corresponding to  $\Psi$ .

It is not difficult to see that the above construction works just as well for  ${}_G\mathcal{M}$  the category of right  $G$ -modules, or  ${}_G\mathcal{M}_G$  the category of  $G$ -bimodules.

Recall that a *right  $G$ -crossed module*  $V$ , is a right  $G$ -module, and a right  $G$ -comodule, such that

$$(v_{(0)} \triangleleft g_{(1)}) \otimes v_{(1)} g_{(2)} = (v g_{(1)})_{(0)} \otimes g_{(2)} (v g_{(1)})_{(1)}, \quad v \in V, g \in G.$$

A very important fact is that for every crossed module  $V$  the map

$$\Psi : V \otimes V \rightarrow V \otimes V, \quad v \otimes w \mapsto w_{(0)} \otimes v \triangleleft w_{(1)}$$

is a braiding for  $V$ . Moreover, in addition to being a right module map, it is also a right comodule map (that is to say it is a morphism in the category  $\mathcal{M}_G^G$  whose objects are right modules and right comodules). Now when  $\Lambda_G^1$  is the cotangent space of a bicovariant calculus, it has a natural right  $G$ -coaction induced by  $\text{Ad}_R$ . Taken in conjunction with its natural right  $G$ -action, this gives  $\Lambda_G^1$  the structure of a right  $G$ -crossed module. We call

the corresponding braiding for  $\Lambda_G^1$  the *bicovariant braiding*. Moreover,  $\Psi$  also induces a braiding for the corresponding calculus  $G \otimes \Lambda_G^1$ : it is given by

$$G \otimes \Lambda_G^{\otimes 2} \rightarrow G \otimes \Lambda_G^{\otimes 2}, \quad g \otimes v \otimes w \mapsto g \otimes \Psi(v \otimes w),$$

where we have tacitly used the isomorphism between  $\Omega^1(G)^{\otimes 2}$  and  $G \otimes \Lambda_G^{\otimes 2}$ . Denoting the corresponding exterior algebras of  $\Lambda_G^1$  and  $\Omega^1(G)$  by  $\Lambda_G^\bullet$  and  $\Omega^\bullet(G)$  respectively, it is easy to see that  $G \otimes \Lambda_G^\bullet$  is isomorphic to  $\Omega^\bullet(G)$ .

### 3 The Quantum Projective Spaces

For  $q \in (0, 1]$  and  $\nu = q - q^{-1}$ , let  $\mathbf{C}_q[M_N]$  be the quotient of the free algebra  $\mathbf{C}\langle u_j^i, | i, j = 1, \dots, N \rangle$  by the ideal generated by the elements

$$\begin{aligned} u_j^i u_j^k - q u_j^k u_j^i, & \quad u_i^j u_k^j - q u_k^j u_i^j, & (1 \leq i < k \leq N); \\ u_j^i u_l^k - u_j^k u_l^i, & \quad u_j^i u_l^k - \nu u_l^i u_j^k, & (1 \leq i < k \leq N, 1 \leq j < l \leq N). \end{aligned}$$

We can put a bialgebra structure on  $\mathbf{C}_q[M_N]$  by introducing a coproduct  $\Delta$  and counit  $\varepsilon$  that act according to  $\Delta(u_j^i) = \sum_{k=1}^N u_k^i \otimes u_j^k$ , and  $\varepsilon(u_j^i) = \delta_{ij}$ . The *quantum determinant* of  $\mathbf{C}_q[M_N]$  is the element

$$\det_N = \sum_{\pi \in S_N} (-q)^{\ell(\pi)} u_{\pi(1)}^1 u_{\pi(2)}^2 \cdots u_{\pi(N)}^N,$$

with summation taken over all permutations  $\pi$  of  $N$  elements, and  $\ell(\pi)$  the length of  $\pi$ . As is well-known,  $\det_N$  is a central and grouplike element of the bialgebra. The centrality of  $\det_N$  makes it easy to adjoin an inverse  $\det_N^{-1}$ . We extend  $\Delta$  and  $\varepsilon$  by setting  $\Delta(\det_N^{-1}) = \det_N^{-1} \otimes \det_N^{-1}$ , and  $\varepsilon(\det_N^{-1}) = 1$ , and denote the new bialgebra by  $\mathbf{C}_q[GL_N]$ . If we assume that  $q$  is real, then we can endow  $\mathbf{C}_q[GL_N]$  with a  $*$ -algebra structure by defining

$$(\det_N^{-1})^* = \det_N, \quad (u_j^i)^* = (-q)^{j-i} \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u_{\pi(l_1)}^{k_1} u_{\pi(l_2)}^{k_2} \cdots u_{\pi(l_{N-1})}^{k_{N-1}},$$

where  $\{k_1, \dots, k_{N-1}\} = \{1, \dots, N\} \setminus \{i\}$  and  $\{l_1, \dots, l_{N-1}\} = \{1, \dots, N\} \setminus \{j\}$  as ordered sets. Moreover, we can give  $\mathbf{C}_q[GL_N]$  a Hopf  $*$ -algebra structure by setting  $S(\det_N^{-1}) = \det_N$ , and  $S(u_j^i) = \det_N^{-1} (u_i^j)^*$ . We denote this Hopf  $*$ -algebra by  $\mathbf{C}_q[U_N]$ . For  $N = 1$ , we get the Hopf algebra  $\mathbf{C}[U_1]$ , where it is usual to denote  $u_1^1 = t$ , and  $\det_N^{-1} = t^{-1}$ . If we quotient  $\mathbf{C}_q[U_N]$  by the ideal  $\langle \det_N - 1 \rangle$ , then the resulting algebra is again a Hopf  $*$ -algebra. We denote it by  $\mathbf{C}_q[SU_N]$ . For  $N = 2$ , we get the well-known algebra  $\mathbf{C}_q[SU_2]$ . We usually denote its four generators by  $a = u_1^1, b = u_2^1, c = u_1^2, d = u_2^2$ .

#### 3.1 The Quantum Projective Spaces

We are now ready to introduce the quantum projective spaces, which are specific examples of quantum flag manifolds. We use a description, introduced in [38], that



presents quantum projective  $N$ -space as a coinvariant subalgebra of  $\mathbf{C}_q[U_N]$  for a right  $\mathbf{C}_q[U_1] \otimes \mathbf{C}_q[U_{N-1}]$ -coaction. This subalgebra is a  $q$ -deformation of the coordinate algebra of the complex manifold  $U_N/(U_1 \times U_{N-1})$ . Recall that classically  $\mathbf{CP}^{N-1}$  is isomorphic to  $U_N/(U_1 \times U_{N-1})$ .

**Definition 3.1.** Let  $\alpha : \mathbf{C}_q[U_N] \rightarrow \mathbf{C}_q[U_{N-1}]$  be the unique Hopf algebra map for which  $\alpha_k(u_j^i) = u_{j-k}^{i-k}$ , if  $i, j > 1$ , and  $\alpha_k(u_j^i) = \delta_{ij}1$  otherwise. Moreover, let  $\hat{\alpha} : \mathbf{C}_q[U_N] \rightarrow \mathbf{C}_q[U_1]$  be the unique Hopf algebra map for which  $\hat{\alpha}_k(u_j^i) = u_j^i$ , if  $i, j < N$ , and  $\hat{\alpha}_k(u_j^i) = \delta_{ij}1$  otherwise. Using these two maps, define a homogeneous right  $\mathbf{C}_q[U_1] \otimes \mathbf{C}_q[U_{N-1}]$ -coaction  $\Delta_{U_N, \alpha}$  on  $\mathbf{C}_q[U_N]$  by

$$\Delta_{U_N, \alpha} = (\text{id} \otimes \hat{\alpha} \otimes \alpha) \circ (\Delta \otimes \Delta) \circ \Delta. \quad (7)$$

Quantum projective  $N$ -space  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$  is defined to be the coinvariant subalgebra of  $\Delta_{U_N, \alpha}$ , that is,

$$\mathbf{C}_q[\mathbf{CP}^{N-1}] = \{f \in \mathbf{C}_q[U_N] \mid \Delta_{U_N, \alpha}(f) = f \otimes 1\}.$$

With a view to finding a set of generators for  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$ , consider the elements  $z_{ij} = u_1^i S(u_j^1)$ , for  $i, j = 1, \dots, N$ . Acting on  $z_{ij}$  by  $\Delta_{U_N, \alpha}$  gives

$$\begin{aligned} \Delta_{U_N, \alpha}(z_{ij}) &= \Delta_{U_N, \alpha}(u_1^i S(u_j^1)) = \sum_{a,b,c,d=1}^N u_a^i S(u_j^d) \otimes \hat{\alpha}(u_b^a S(u_c^d)) \otimes \alpha(u_1^b S(u_c^1)) \\ &= \sum_{a,d=1}^N u_a^i S(u_d^1) \otimes \hat{\alpha}(u_1^a S(u_d^1)) \otimes \alpha(u_1^1 S(u_1^1)) \\ &= u_1^i S(u_1^1) \otimes \hat{\alpha}(u_1^1 S(u_1^1)) \otimes 1 \\ &= u_1^i S(u_1^1) \otimes 1 \otimes 1. \end{aligned}$$

Thus,  $z_{ij} \in \mathbf{C}_q[\mathbf{CP}^{N-1}]$ , for all  $i, j = 1, \dots, N$ . To show that the  $z_{ij}$  actually generate  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$  is more difficult, and requires some representation. We refer the interested reader to [28, 44].

As one would expect, the definition of  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$  given here is equivalent to the one used in [40]. There  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$  was presented as an invariant subspace of  $\mathbf{C}_q[SU_N]$ . For a discussion of the relationship between the different constructions see [38].

## 4 A Differential Structure for the Quantum Projective Bundles

In this section we introduce a differential structure for the quantum homogeneous bundle  $\alpha : \mathbf{C}_q[U_N] \rightarrow \mathbf{C}_q[U_1] \otimes \mathbf{C}_q[U_{N-1}]$ . We describe the parallelizable calculi on total space  $\mathbf{C}_q[U_N]$  in terms generators and relations, and give explicit formulae for the right action of the generators of each algebra.

#### 4.1 A Left Covariant First Order Differential Calculus for $\mathbf{C}_q[U_N]$

We will now introduce a bicovariant differential calculus for  $\Omega_q^1(U_N)$  that will be central in our later work. It is different from the standard bicovariant calculus on  $\mathbf{C}_q[U_N]$  discussed in [40], as we shall see below.

**Proposition 4.1** *Denote by  $\Omega_q^1(U_N)$  the left-covariant calculus on  $\mathbf{C}_q[U_N]$  corresponding to the right ideal  $I_{U_N}$  generated by the elements, for  $i, j, k, l = 1, \dots, N; i \neq j, k \neq l$ ,*

$$u_j^i u_l^k, u_l^k (u_i^i - 1), (u_i^i - 1)(u_k^k - 1), u_l^k (\det_N^{-1} - 1), (u_i^i - 1)(\det_N^{-1} - 1). \quad (8)$$

*The cotangent space  $\Lambda_{U_N}^1 = \mathbf{C}_q[U_N]^+ / I_{U_N}$  of  $\Omega^1(U_N)$  has a basis  $e_{ij} = \overline{u_j^i - \delta_{ij}}$ , for  $i, j = 1, \dots, N$ , and right module relations*

$$e_{ij} u_l^k = u_l^k e_{ij}, \quad e_{ii} u_l^k = u_l^k e_{ii}, \quad i \neq j. \quad (9)$$

**Proof.** Let  $\Lambda_{U_N}^1$  be an  $N^2$ -dimensional vector space with basis  $\{e_{ij}\}_{i,j=1}^N$ , and a right  $\mathbf{C}_q[U_N]$ -action  $\triangleleft$  given by, for  $i, j, k, l = 1, \dots, N; k \neq l$ ,

$$e_{ij} \triangleleft u_l^k = 0, \quad e_{ij} \triangleleft u_k^k = e_{ij}, \quad e_{ij} \triangleleft \det_N^* = e_{ij}.$$

We will establish the proposition by constructing a right  $\mathbf{C}_q[U_N]$ -module isomorphism from  $\mathbf{C}_q[U_N]^+ / I$  to  $\Lambda_{U_N}^1$ : Consider the following basis of  $\mathbf{C}_q[M_N]$

$$\left\{ \prod_{i=1}^N \left( \prod_{j=1}^N (u_j^i)^{t_{ij}} \right) \left( \prod_{k=1}^N (u_k^k)^{t_k} \right) (\det^*)^{t_0} \mid t_{ij}, t_k, t_0 \in \mathbb{N}_0 \right\}$$

(For a proof that this is indeed a basis see [42].) This gives a natural basis for  $\mathbf{C}_q[M_N] \otimes \mathbf{C}[\det_N^*]$  (where we have not yet assumed the relation  $\det_N \det_N^* = 1$ .) Define a map  $\varphi : \mathbf{C}_q[M_N] \otimes \mathbf{C}[\det_N^*] \rightarrow \Lambda_{U_N}^1$  by setting, for  $i \neq j$ ,

$$\varphi(u_j^i \prod_{k=1}^N (u_k^k)^{t_k}) = e_{ij}, \quad \varphi(\det^*) = \sum_{k=1}^N e_{kk}, \quad \varphi(1) = \frac{1}{2} \sum_{k=1}^N e_{kk},$$

defining recursively

$$\varphi(u_i^i) = e_{ii} + \varphi(1), \quad \varphi(u_i^i \prod_{k=i}^N (u_k^k)^{t_k}) = e_{kk} + \varphi(\prod_{k=i}^N (u_k^k)^{t_k}),$$

and finally sending all other basis elements to zero. That  $\varphi$  descends to a map from  $\mathbf{C}_q[U_N]$  to  $V$  follows from the fact that

$$\varphi(\det_N \det^* - 1) = \varphi\left(\left(\prod_{i=1}^N u_i^i\right) \det_N^*\right) - \sum_{i=1}^N e_{ii} = \sum_{i=1}^N e_{ii} - \sum_{i=1}^N e_{ii} = 0.$$

Let us now show that  $\varphi(I) = 0$ : That  $\varphi(u_j^i u_l^k f) = 0$ , for all  $f \in \mathbf{C}_q[U_N]$  is clear from the definition of  $\varphi$ . That  $\varphi((u_i^i - 1)u_l^k) = 0$  follows from

$$\varphi(u_l^k(u_i^i - 1)) = \varphi(u_l^k u_i^i) - \varphi(u_l^k) = e_{kl} - e_{kl} = 0.$$

An analogous calculation will establish that  $\varphi(u_l^k(u_i^i - 1)u_q^p) = 0$ , for all  $p, q = 1, \dots, N$ , implying that  $\varphi((u_i^i - 1)u_l^k f) = 0$ , for all  $f \in \mathbf{C}_q[U_N]$ . That  $\varphi((u_i^i - 1)(u_j^j - 1)) = 0$  follows from

$$\begin{aligned} \varphi((u_i^i - 1)(u_j^j - 1)) &= \varphi(u_i^i u_j^j) - \varphi(u_i^i) - \varphi(u_j^j) + \varphi(1) \\ &= e_{ii} + e_{jj} + \varphi(1) - e_{ii} - \varphi(1) - e_{jj} - \varphi(1) + \varphi(1) \\ &= 0. \end{aligned}$$

Again an analogous calculation will establish that  $\varphi((u_i^i - 1)(u_j^j - 1)u_q^p) = 0$ , for all  $p, q = 1, \dots, N$ , implying that  $\varphi((u_i^i - 1)(u_j^j - 1)f) = 0$ , for all  $f \in \mathbf{C}_q[U_N]$ . Finally, from the definition of  $\varphi$ , it is obvious that  $(\det_N^* - 1)f$  is contained in  $\ker(\varphi)$ , for all  $f \in \mathbf{C}_q[U_N]$ . This gives us that  $I \subseteq \ker(\varphi)$ , and so, the dimension of  $\mathbf{C}_q[U_N]^+/I$  must be greater than or equal to  $N^2$ . However, it is clear from the given generators of  $I$ , that  $\mathbf{C}_q[U_N]^+/I$  is spanned by the elements  $\overline{u_j^i - \delta_{ij}}$ , and so, must have dimension less than or equal to  $N^2$ . We can conclude that  $\Lambda_{U_N}^1$  and  $\mathbf{C}_q[U_N]^+/I$  are isomorphic as vector spaces. One shows that the two are isomorphic as right  $\mathbf{C}_q[U_N]$ -modules by comparing the right actions of the generators of  $\mathbf{C}_q[U_N]$  on the basis of each space. This is trivial except for the case of  $u_k^k$  acting on  $\overline{u_j^i}$ . For  $i \neq j$ , we have

$$\varphi(\overline{u_j^i} \triangleleft u_k^k) = \varphi(\overline{u_j^i u_k^k}) = e_{ij} = e_{ij} \triangleleft u_k^k = \varphi(\overline{u_j^i}) \triangleleft u_k^k,$$

as required. It is similarly shown that  $\varphi(\overline{u_i^i} \triangleleft u_k^k) = \varphi(\overline{u_i^i}) \triangleleft u_k^k$ . The right module relations of the calculus now follow directly: For example

$$e_{ii} u_l^k = \sum_{a=1}^N u_a^k (e_{ii} \triangleleft u_a^a) = u_l^k (e_{ii} \triangleleft u_l^l) = u_l^k (e_{ii}).$$

□

Let us now calculate the action of the exterior derivative on the generators of  $\mathbf{C}_q[U_N]$ :

$$d(\det_N^*) = \det_N^* \otimes \det_N^* - \det_N^* \otimes 1 = \det_N^* \otimes \overline{\det_N^* - 1} = \frac{1}{2} \sum_{k=1}^N \det_N^* e_{kk}.$$

and

$$du_j^i = \left( \sum_{k=1}^N u_k^i \otimes \overline{u_j^k} \right) - u_j^i \otimes 1 = \sum_{k=1}^N u_k^i e_{kl}.$$

We now give explicit relations for the right action of the generators on the exterior derivative of the generators:

$$(du_j^i)u_q^p = \sum_{k,l=1}^N u_k^i e_{kl} u_q^p = \sum_{k,l=1}^N u_k^i u_q^p e_{ij} = \sum_{k,l,m=1}^N u_k^i u_q^p S(u_m^i) du_j^m.$$

## 4.2 The Induced Calculus on $\mathbf{C}_q[\mathbf{CP}^{N-1}]$

We would now like to use Theorem 2.1 to describe the calculus induced on  $\mathbf{C}_q[\mathbf{CP}^{N-1}]$  by restriction. We begin by establishing a useful lemma:

**Lemma 4.2** *In  $\Lambda_{U_N}^1$ , it holds that*

$$\overline{S(u_j^i)} = (-q)^{i-j+|i-j|-1} e_{ij}.$$

Moreover, for  $i \neq j$ , it also holds that

$$\bar{v} \triangleleft S(u_j^i) = 0, \quad \bar{v} \triangleleft S(u_i^i) = \bar{v}, \quad v \in \mathbf{C}_q[U_N]^+.$$

**Proof.** For  $i \neq j$ , and the ordered sets  $\{k_1, \dots, k_{N-1}\} = \{1, \dots, N\} \setminus \{j\}$ , and  $\{l_1, \dots, l_{N-1}\} = \{1, \dots, N\} \setminus \{i\}$ , we have

$$\overline{S(u_j^i)} = \overline{(u_i^j)^* \det_N^{-1}} = (-q)^{i-j} \left( \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} \overline{u_{\pi(l_1)}^{k_1} u_{\pi(l_2)}^{k_2} \cdots u_{\pi(l_{N-1})}^{k_{N-1}}} \right) \overline{\det_N^{-1}}.$$

Now for any  $\pi \in S_{N-1}$ , such that  $k_m \neq \pi(l_m)$ , for more than one  $m = 1, \dots, N-1$ , it must hold that

$$\overline{u_{\pi(l_1)}^{k_1} u_{\pi(l_2)}^{k_2} \cdots u_{\pi(l_{N-1})}^{k_{N-1}}} = 0.$$

The only permutation for which this does not happen is  $\pi_{ij} = (i, i+1) \cdots (j-2, j-1)$ . Thus, we have that

$$\begin{aligned} \overline{S(u_j^i)} &= (-q)^{i-j} (-q)^{\ell(\pi_{ij})} \overline{u_{\pi_{ij}(l_1)}^{k_1} u_{\pi_{ij}(l_2)}^{k_2} \cdots u_{\pi_{ij}(l_{N-1})}^{k_{N-1}}} \overline{\det_N^{-1}} \\ &= (-q)^{i-j} (-q)^{|i-j|-1} \overline{u_{\pi_{ij}(l_1)}^{k_1} u_{\pi_{ij}(l_2)}^{k_2} \cdots u_{\pi_{ij}(l_{N-1})}^{k_{N-1}}} \overline{\det_N^{-1}} \\ &= (-q)^{i-j} (-q)^{|i-j|-1} \overline{u_j^i} = (-q)^{i-j} (-q)^{|i-j|-1} e_{ij}. \end{aligned}$$

We now turn to the action of  $S(u_j^j)$  on  $e_{ij}$ : For the ordered sets  $\{k_1, \dots, k_{N-1}\} = \{1, \dots, N\} \setminus \{s\}$ , and  $\{l_1, \dots, l_{N-1}\} = \{1, \dots, N\} \setminus \{r\}$ , we have

$$e_{ij} \triangleleft S(u_s^r) = (-q)^{r-s} \left( \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} e_{ij} \triangleleft \overline{u_{\pi(l_1)}^{k_1} u_{\pi(l_2)}^{k_2} \cdots u_{\pi(l_{N-1})}^{k_{N-1}}} \right) \overline{\det_N^{-1}}.$$

Now for any  $\pi \in S_{N-1}$ , such that  $k_m \neq \pi(l_m)$ , for any  $m = 1, \dots, N-1$ , it must hold that

$$e_{ij} \triangleleft \overline{u_{\pi(l_1)}^{k_1} u_{\pi(l_2)}^{k_2} \cdots u_{\pi(l_{N-1})}^{k_{N-1}}} = 0.$$

If  $r \neq s$ , then there is no permutation for which this does not happen, and so, we get  $e_{ij} \triangleleft S(u_s^r) = 0$ . For  $r = s$ , the identity is the only permutation for which this not occur. Thus,

$$e_{ij} \triangleleft S(u_r^r) = (-q)^{r-r} (-q^0) e_{ij} \triangleleft (u_{k_1}^{k_1} \cdots u_{k_{N-1}}^{k_{N-1}}) = e_{ij}.$$

□

Let us now find a basis for  $V_{\mathbf{C}P^{N-1}} = \mathbf{C}_q[\mathbf{C}P^{N-1}]^+ / (I_{U_N} \cap \mathbf{C}_q[\mathbf{C}P^{N-1}])$ :

**Lemma 4.3** *The set  $\{\overline{z_{i1}}, \overline{z_{1i}} \mid i = 1, \dots, N\}$  forms a basis for  $V_{\mathbf{C}P^{N-1}}$  of dimension  $2(N-1)$ .*

**Proof.** The proof here is routine: One identifies  $V_{\mathbf{C}P^{N-1}}$  with its canonical image in  $\Lambda_{U_N}^1$ , and then uses the previous lemma to show that  $\overline{z_{i1}} = e_i^+$ , that  $\overline{z_i} = e_i^-$ , and  $\overline{z_{ij}} = 0$  for  $i, j > r$ . It remains to show that  $e_{11}, e_{kl} \notin V_{\mathbf{C}P^{N-1}}$ , for  $k, l \neq 1$ : From the proof of Proposition 4.1, is clear that

$$\overline{z_{11} - 1} = \overline{u_1^1 S(u_1^1) - 1} = \sum_{i=1}^N e_{ii} - \sum_{i=1}^N e_{ii} = 0.$$

That  $\overline{z_{ij}} = \overline{u_1^i S(u_j^1)} = 0$ , for all  $i, j \geq 2$ , follows from the lemma above. The fact that the subspace spanned by the elements  $e_{i1}, e_{1i}$ , for  $i = 1, \dots, N$ , is closed under the right action of  $\mathbf{C}_q[\mathbf{C}P^{N-1}]$ , now implies that it is equal to  $V_{\mathbf{C}P^{N-1}}$ . □

As an easy corollary we get:

**Corollary 4.4** *It holds that  $I_{\mathbf{C}P^{N-1}}$  is generated as a right ideal of  $\mathbf{C}_q[\mathbf{C}P^{N-1}]^+$  by the elements, for  $i \neq j, k \neq l, r \neq s$ ,*

$$z_{11} - 1; \quad z_{ij} z_{kl}, \quad i, j, k, l = 1, \dots, N; \quad z_{rs}, \quad r, s \geq 2. \quad (10)$$

Let us now show that the coaction  $\Delta_{\mathbf{C}P^{N-1}}$  is well-defined on  $V_{\mathbf{C}P^{N-1}}$ .

**Lemma 4.5** *It holds that  $\Delta_{\mathbf{C}P^{N-1}}$  is well-defined on  $V_{\mathbf{C}P^{N-1}}$*

**Proof.** We need to show that  $\Delta_{\mathbf{C}P^{N-1}}$  carries each of the generators of  $I_{\mathbf{C}P^{N-1}}$  given in (10) into  $I_{\mathbf{C}P^{N-1}} \otimes \mathbf{C}_q[U_{N-1}] \otimes \mathbf{C}_q[1]$ . Let us look at  $z_{rs}$ , for  $r, s \geq 2$ :

$$\Delta_{\mathbf{C}P^{N-1}}(z_{rs}) = \Delta_{\mathbf{C}P^{N-1}}(u_1^r S(u_s^1)) = \sum_{a,b,c,d=1}^N u_1^b S(u_c^1) \otimes S(\widehat{a}(u_b^a S(u_d^c))) \otimes \alpha(u_a^r S(u_s^d)).$$

Now when  $a = 1$ , or  $d = 1$ , we have  $\alpha(u_a^r S(u_s^d)) = 0$ , and so, we must have that  $\Delta_{\mathbf{C}P^{N-1}}(z_{rs})$  is contained in  $I_{\mathbf{C}P^{N-1}} \otimes \mathbf{C}_q[U_{N-1}] \otimes \mathbf{C}_q[1]$ .

That the same holds true for the other generators is similarly established.  $\square$

Let us now look at the framing of  $\Omega_q^1(\mathbf{C}P^{N-1})$  given by Theorem 2.1. Explicitly, the action of the soldering form  $\theta$  on the basis elements just produced is given by

$$\theta(\overline{z_{ij}}) = m \circ (S \otimes d) \left( \sum_{k,l=1}^N u_k^i S(u_1^l) \otimes u_1^k S(u_l^1) \right) = \sum_{k,l=1}^N q^{2(l-1)} u_1^l S(u_k^i) dz_{kl}; \quad (11)$$

$$\theta(\overline{z_{ji}}) = m \circ (S \otimes d) \left( \sum_{k,l=1}^N u_k^1 S(u_i^l) \otimes u_1^k S(u_l^1) \right) = \sum_{k,l=1}^N q^{2(l-i)} u_i^l S(u_k^1) dz_{kl}. \quad (12)$$

We defer deriving explicit formulae for the action of the exterior derivative to the section discussing complex structures.

## 5 Framing the Tensor Powers of A Base Space Calculus

In this section we will construct a framing for tensor powers of the base space calculus of a quantum homogeneous bundle. This tool will be of central importance in later sections. We begin by introducing a natural condition required for our framing result to work: Let  $\pi : G \rightarrow H$  be a quantum homogeneous bundle with base space  $M$ , and let  $\Omega^1(G)$  be a calculus on  $G$  with cotangent space  $\Lambda_G^1$ . Considering  $V_M$  as embedded in  $\Lambda_G^1$ , we say that the restriction of  $\Omega^1(G)$  to  $M$  is *right-absorbing* if  $V_M G \subseteq V_M$ . We will now look at some standard examples and test whether they are right-absorbing or not.

**Example 5.1.** We begin with the bundle  $(\mathbf{C}_q[SU_N], S^{2N-1}, I_{SU_N})$  described in [40]. Since  $\Lambda_{SU_N}^1 = V_{S^{2N-1}}$  it is clear that the bundle is right absorbing.

For the quantum  $(N-1)$ -complex projective bundle  $(SU_N, \mathbf{C}_q[\mathbf{C}P^{N-1}], I_{SU_N})$  also described in [40], we have

$$\ker(\overline{\alpha_N}) = \text{span}_{\mathbf{C}} \{e_i^+, e_i^- | i = 1, \dots, N\} = V_{\mathbf{C}P^{N-1}}.$$

Thus, this bundle is also right-absorbing. The same is true for  $\mathbf{C}_q[\mathbf{C}P^{N-1}]$  presented as the base of the bundle  $\hat{\alpha} \otimes \alpha : \mathbf{C}_q[U_N] \rightarrow \mathbf{C}_q[U_1] \otimes \mathbf{C}_q[U_{N-1}]$ , with respect to the differential calculus introduced above.

It is easy to show that the differential structures induced on the bundles  $\alpha_N : \mathbf{C}_q[SU_N] \rightarrow \mathbf{C}_q[U_{N-1}]$ , and  $\beta_N : \mathbf{C}_q[SU_N] \rightarrow \mathbf{C}_q[SU_{N-1}]$ , are also right-absorbing.

To find an example where the condition does not hold, we turn to the universal calculus for the quantum  $(2N-1)$ -sphere  $\mathbf{C}_q[S^{2N-1}]$ . A routine investigation will show that it is not right absorbing.

With the notion of right-absorbing in hand, we are now ready to prove the promised framing result.

**Theorem 5.2** *Let  $\pi : G \rightarrow H$  be a right-absorbing quantum homogeneous bundle satisfying the properties of Theorem 2.1. The right  $H$ -comodule  $(V_M)^{\otimes k}$ , with the tensor product coaction  $\Delta_M^{\otimes k}$ , frames the tensor product  $(\Omega^1(M))^{\otimes k}$ .*

**Proof.** Consider the map

$$s_{\theta,k} := s_{\theta} \otimes (\theta^{\otimes k-1}) : G \otimes (\Lambda_G^1)^{\otimes k} \rightarrow (\Omega^1(G))^{\otimes k},$$

and note that it acts according to

$$f \otimes \overline{v_1} \otimes \cdots \otimes \overline{v_k} \mapsto f(v_1)_{(1)} d(v_1)_{(2)} \otimes_G \cdots \otimes_G (v_k)_{(1)} d(v_k)_{(2)}.$$

This map is an isomorphism, as we will show by constructing its inverse: Since every element of  $(G \otimes \Lambda_G^1)^{\otimes k}$  can be written in the form  $(f \otimes \overline{v_1}) \otimes_G (1 \otimes \overline{v_2}) \otimes_G \cdots \otimes_G (1 \otimes \overline{v_k})$ , for some unique  $f \in G$ ,  $v_i \in G$ ,  $i = 1, \dots, k$ , we have a canonical isomorphism

$$c : (G \otimes \Lambda_G^1)^{\otimes k} \rightarrow G \otimes (\Lambda_G^1)^{\otimes k}.$$

Composing this map with  $(s_{\theta}^{-1})^{\otimes k} : \Omega^1(G)^{\otimes k} \rightarrow (G \otimes \Lambda_G^1)^{\otimes k}$  gives a map from  $\Omega^1(G)^{\otimes k}$  to  $G \otimes (\Lambda_G^1)^{\otimes k}$ , which is easily seen to act as an inverse to  $s_{\theta,k}$ . Moreover, if we endow  $\Omega^1(G)^{\otimes k}$  with the tensor product coaction  $\Delta_R^{\otimes k}$ , and  $G \otimes (\Lambda_G^1)^{\otimes k}$  with the tensor product coaction  $\Delta_{G,\pi} \otimes (\text{Ad}_R)^{\otimes k}$ , then  $s_{\theta,k}$  is a right  $H$ -comodule map.

Now as a little careful thought will confirm, the image of  $(\Omega^1(M))^{\otimes k}$  in  $(\Omega^1(G))^{\otimes k}$  is equal to  $(\Omega^1(M))^{\otimes k}$  (note that we have one tensor product over  $M$  and another over  $G$ ). Moreover, if we endow  $G \otimes V_M^{\otimes k}$  with the tensor product  $\Delta_{G,\pi} \otimes (\Delta_M)$ , then since  $\Delta_M$  is equal to the restriction of  $\text{Ad}_R$  to  $M^+$ , the embedding  $G \otimes V_M^{\otimes k} \hookrightarrow G \otimes (\Lambda_G^1)^{\otimes k}$  is a right  $H$ -comodule map.

Now for the theorem to be true, the image of  $\Omega^1(M)^{\otimes k}$  in  $\Omega^1(G)^{\otimes k}$  must lie in the image  $G \otimes V_M^{\otimes k}$  in  $G \otimes (\Lambda_G^1)^{\otimes k}$ . To see that this is so, consider the general element

$$m_0 dm_1 \otimes_G \cdots \otimes_G dm_k \in (\Omega^1(M))^{\otimes k}.$$

Its image under  $s_k$  is given by

$$c((m_0(m_1)_{(1)} \otimes \overline{(m_1)_{(2)}}) \otimes_G \cdots \otimes_G (m_k)_{(1)} \otimes \overline{(m_k)_{(2)}}).$$

Since  $\Delta(m) \in G \otimes M$ , for all  $m \in M$ , and  $V_M$  is closed under the right action of  $G$ , it is clear that  $s_k(m_0 dm_1 \otimes_G \cdots \otimes_G dm_k)$  is contained in  $G \otimes V_M^{\otimes k}$ .

For the sake of clarity, we present these facts in the following commutative diagram:

$$\begin{array}{ccc} (\Omega^1(G)^{\otimes k}, \Delta_R^{\otimes k}) & \xrightarrow{s_k^{-1}} & (G \otimes (\Lambda_G^1)^{\otimes k}, \Delta_{\pi} \otimes \Delta_R^{\otimes k}) \\ \uparrow & & \uparrow \\ (\Omega^1(M)^{\otimes k}, \Delta_{\text{id}}) & \xrightarrow{s_k^{-1}} & ((G \otimes V_M^{\otimes k})^H, \Delta_{\pi} \otimes \Delta_M^{\otimes k}), \end{array}$$

where  $\Delta_{\text{id}}$  is the trivial coaction that acts as  $\Delta_{\text{id}}(\omega) = \omega \otimes 1$ , for all  $\omega \in (\Omega^1(M))^{\otimes_M k}$ . It is clear that the image of  $\Omega^1(M)^{\otimes_M k}$  under  $s_k^{-1}$  is coinvariant under the coaction  $\Delta_M^{\otimes k}$ . Thus, to show that the result is true, it remains to show that every element of  $G \otimes V_M^{\otimes k}$  coinvariant under  $\Delta_\pi \otimes \Delta_M^{\otimes k}$  is contained in the image of  $\Omega^1(M)^{\otimes_M k}$  under  $s_k^{-1}$ . To this end, consider the element

$$\sum_i f^i \otimes m_1^i \otimes m_k^i \in (G \otimes V_M^{\otimes k})^H$$

Its image under  $s_k$  is given by

$$\sum_i f^i(m_1^i)_{(1)} d(m_1^i)_{(2)} \otimes_G \cdots \otimes_G (m_k^i)_{(1)} d(m_k^i)_{(2)}$$

Since  $\Omega^1(M)G \subseteq G\Omega^1(M)$ , it is clear that this is contained in  $G\Omega^1(M) \otimes_G \cdots \otimes_G \Omega^1(M)$ . Moreover, since it is the image of an element coinvariant under  $\Delta_\pi \otimes \Delta_M^{\otimes k}$ , it must lie in  $\Omega^1(M) \otimes_G \cdots \otimes_G \Omega^1(M)$ .  $\square$

**Example 5.3.** Woronowicz's well-known 3D calculus [49] over  $\mathbf{C}_q[SU(2)]$  corresponds to the ideal generated by the elements

$$a + q^{-2}d - (1 + q^{-2}), \quad bc, \quad b^2, \quad c^2, \quad (a-1)b, \quad (a-1)c.$$

Its module of left-invariant forms has a basis  $e^0 = \overline{u_1^1 - 1}, e^+ = \overline{u_1^2}, e^- = \overline{u_2^1}$ , and  $V_{\mathbf{C}P^1} = \text{span}_{\mathbf{C}}\{e^+, e^-\}$ . It is easily seen to be right-absorbing.

For the framing, we first we note that

$$(V_{\mathbf{C}P^{N-1}})^{\otimes 2} = \text{span}_{\mathbf{C}}\{e^\pm \otimes e^\pm, e^\pm \otimes e^\mp\}.$$

It is easy to see that

$$\Delta_{\mathbf{C}P^{1,2}}(e^\pm \otimes e^\pm) = e^\pm \otimes e^\pm \otimes t^\mp, \quad \Delta_{\mathbf{C}P^{1,2}}(e^\pm \otimes e^\mp) = e^\pm \otimes e^\mp \otimes 1.$$

Thus we have

$$(\Omega_u^1(M))^{\otimes 2} \simeq \mathcal{E}_2 \otimes (M^+)^{\otimes 2} \otimes \mathcal{E}_{-2}.$$

$\square$

It is natural to ask if the right-absorbing condition is necessary for the framing result to hold. However, it is quite easy to see that it is: The image of  $du_1^i \otimes du_1^k$  under  $s_{\theta,k}$  is given by  $u_k^i u_a^j \otimes u_1^k u_l^a \otimes u_1^l$ . Clearly this is not contained in  $\mathbf{C}_q[SU_N] \otimes (\mathbf{C}_q[S^{2N-1}]^+)^{\otimes k}$ . Now while  $s_k((\Omega^1(M))^{\otimes_M k})$  is not contained in  $G \otimes V_M^{\otimes k}$ , it is contained in  $G \otimes \ker(\bar{\pi})^{\otimes k} \otimes V_M$  (this follows from the fact that  $G\Omega^1(M)G = \ker(\text{ver})$ ). Thus, one might ask if  $\ker(\bar{\pi})^{\otimes k} \otimes V_M$ , with the right coaction  $\text{Ad}_R^{\otimes(k-1)} \otimes \Delta_M$ , frames  $(\Omega^1(M))^{\otimes_M k}$ . The following example shows that this is not the case:



**Example 5.4.** Consider the surjective Hopf algebra map  $\gamma_N : \mathbf{C}_q[SU_N] \rightarrow \mathbf{C}[U_1]$  by setting  $\gamma_N(u_1^1) = t^{-1}$ ;  $\gamma_N(u_k^k) = 1$ , for  $k = 2, \dots, N-1$ ;  $\gamma_N(u_N^N) = t$ ; and  $\gamma_N(u_j^i) = 0$ , for  $i, j = 1, \dots, N$ , and  $i \neq j$ . We denote the associated right  $\mathbf{C}_q[U_1]$ -coaction by  $\Delta_{U(N), \gamma_N}$ . We call the base space the *quantum complex  $(N, N-1)$ -Stiefel manifold* and denote it by  $\mathbf{C}_q[S^{N, N-1}]$  (see [44] for further details on general quantum Steifel manifolds).

For  $N \geq 4$ , routine calculation will show that, with  $p \neq q$ , and  $p, q = 2, \dots, N-1$ , we have

$$u_N^1 \otimes u_1^N \otimes u_q^p \in (\ker(\gamma_N))^{\otimes 2} \otimes (\mathbf{C}_q[S^{N, N-1}]).$$

Moreover, it is easy to show that

$$\Delta_M^{\otimes k}(u_N^1 \otimes u_1^N \otimes u_q^p) = u_N^1 \otimes u_1^N \otimes u_q^p \otimes 1,$$

giving us that

$$1 \otimes u_N^1 \otimes u_1^N \otimes u_q^p \in (\mathbf{C}_q[U_N] \otimes (\ker(\gamma_N))^{\otimes 2} \otimes (\mathbf{C}_q[S^{N, N-1}]))^{U_1}.$$

However, as is easy to see,

$$s \otimes (s_\theta)^{\otimes 2} (1 \otimes u_N^1 \otimes u_1^N \otimes u_q^p) \notin (\Omega_u^1(S^{N, N-1}))^{\otimes 2}.$$

Thus,  $(\Omega_u^1(S^{N, N-1}))^{\otimes k}$  is not framed by  $(\ker(\gamma_N))^{\otimes k-1} \otimes M^+$ .  $\square$

## 6 Woronowicz Braidings, Exterior Algebras, and Framings

We would now like to construct an exterior algebra for  $\Omega_q^1(U_N)$  using Woronowicz's braiding approach, as discussed in the introduction. Since  $\Omega_q^1(U_N)$  is not bicovariant, it is not immediately clear that there exists a natural braiding for us to use. However, as we will see below, the construction of the bicovariant braiding can be applied to a more general type of calculus. Moreover, the braiding produced commutes with the framing coaction. Crucially, this generalised construction can be successfully applied to  $\Omega^1(U_N)$ .

### 6.1 Woronowicz Braidings

Let  $G$  be a Hopf algebra,  $I_G$  a right ideal of  $G^+$ , and  $\Lambda_G^1 = G^+/I_G$  the cotangent space of the corresponding calculus. Denoting the space of linear endomorphisms of  $\Lambda_G^1$  by  $\text{End}(\Lambda_G^1)$ , we have a map

$$\triangleleft : G \rightarrow \text{End}(\Lambda_G^1), \quad g \rightarrow \triangleleft g,$$

where  $\triangleleft g$  acts on  $\Lambda_G^1$  in the obvious way. Now if it holds that

$$(\text{id} \otimes \triangleleft) \circ \text{Ad}_R(I_G) \subseteq I_G \otimes G + \Lambda_G^1 \otimes \ker(\triangleleft), \quad (13)$$

then we have a well-defined linear map

$$\Psi : \Lambda_G^1 \otimes \Lambda_G^1 \rightarrow \Lambda_G^1 \otimes \Lambda_G^1, \quad \overline{v} \otimes \overline{w} \rightarrow \overline{w_{(0)}} \otimes \overline{v} \triangleleft w_{(1)}. \quad (14)$$

Moreover, since  $\Psi$  is clearly induced on  $\Lambda_G^1 \otimes \Lambda_G^1$  by the bicovariant braiding for the universal calculus, it will commute with the right action of  $G$ , and satisfy the braid relation. Thus, we can use it to construct an exterior algebra for  $G$ . This motivates the following definition:

**Definition 6.1.** Let  $G$  be a Hopf algebra, and  $\Omega^1(G)$  a left-covariant calculus for  $G$  with corresponding right ideal  $I_G$ . If (13) holds, then we call the braiding defined by (14) the *Woronowicz braiding* of  $\Omega^1(G)$ . Moreover, we say that  $\Omega^1(G)$  *admits a Woronowicz braiding*.

## 6.2 Woronowicz Braidings and Framings

Since  $\Delta(M) \subseteq G \otimes M$ , we have

$$\text{Ad}_R(v) = v_{(2)} \otimes \pi(S(v_{(1)}))\pi(v_{(3)}) = v_{(2)} \otimes \pi(S(v_{(1)})).$$

Thus,  $\Delta_M$  coincides with  $\text{Ad}_R$  on  $M^+$ . Moreover, if  $\Psi_u$  is the bicovariant braiding for the universal calculus, then

$$(\Psi_u \otimes \text{id}) \circ \Delta_M^{\otimes 2} = \Delta_M^{\otimes 2} \circ \Psi_u. \quad (15)$$

Now take a non-universal calculus on  $G$  with cotangent space  $\Lambda_G^1$ , admitting a Woronowicz braiding  $\Psi$ , and for which  $\Delta_M(I_M) \subseteq I_M \otimes H$ . Since  $\Psi$  is induced on  $\Lambda_G^1 \otimes \Lambda_G^1$  by  $\Psi_u$ , it follows from (15) that  $(\text{id} \otimes \Psi) \circ \Delta_M^{\otimes 2} = \Delta_M^{\otimes 2} \circ \Psi$ . This in turn gives implies that

$$((\Psi + \text{id}) \otimes \text{id}) \circ \Delta_M^{\otimes 2} = \Delta_M^{\otimes 2} \circ (\Psi + \text{id}).$$

Thus, we see that  $(\Delta_M)^{\otimes k}$  induces a well-defined map on the quotient  $V_M^k = (V_M)^{\otimes k} / \ker(A_{\Psi,k})$ . This is obviously an important property since  $V_M^k$  will usually be a much simpler object than  $V_M^{\otimes k}$ . We collect these facts in the following lemma:

**Lemma 6.2** *Let  $\Omega^1(G)$  be a left-covariant calculus on a Hopf algebra  $G$  with corresponding right ideal  $I_G$ , admitting a Woronowicz braiding  $\Psi$ . Moreover, let  $M$  be the base of a quantum  $G$ -homogeneous space  $\pi : G \rightarrow H$ , for which  $\Delta_M(I_M) \subseteq I_M \otimes H$ . The coaction  $(\Delta_M)^{\otimes k}$  induces a well-defined  $H$ -coaction on  $V_M^k = (V_M)^{\otimes k} / \ker(A_{\Psi,k})$ .*

## 6.3 Total Differential Calculi

What we lack for a total differential calculus is an exterior derivative. The following lemma shows that one exists and that it is necessarily unique. The proof used is a variation on the original proof of Woronowicz [49]. We note that in general it cannot be used to show that the exterior algebra of  $\Omega^1(G)$  admits an exterior derivative.

**Lemma 6.3** *Let  $G$  be Hopf algebra, and  $(\Omega^1(G), d)$  a first-order left-covariant calculus for  $G$  which admits a Woronowicz braiding  $\Psi$ . Moreover, let  $\pi : G \rightarrow H$ , be a quantum homogeneous space with coinvariant subspace  $M$ . Denoting the exterior algebra extending  $\Omega^1(M)$  by  $\Omega^\bullet(M)$ , there exists a unique extension of  $d$  to a linear map on  $\Omega^\bullet(M)$  such that  $(\Omega^1(M), d)$  is a total differential calculus.*

**Proof.** Let  $\mathbf{C}X$  be the one-dimensional complex vector space spanned by  $X$ , and denote

$$\widetilde{V}_M = V_M \oplus \mathbf{C}X.$$

We extend the  $G$ -module structure of  $V_M$ , to a  $G$ -module structure of  $\widetilde{V}_M$ , by setting

$$(\bar{v} + \lambda X)g = \bar{v} \triangleleft g + \lambda \varepsilon(g)X + (g - \varepsilon(g)), \quad g \in G, \lambda \in \mathbf{C}. \quad (16)$$

We extend the  $G$ -comodule structure of  $V_M$ , to a  $G$ -comodule structure of  $\widetilde{V}_M$ , by setting

$$\bar{v} + \lambda X \mapsto \Delta_M(\bar{v}) + \lambda X \otimes 1.$$

It is easy to check that, with respect to these structures,  $V_M$  is a right crossed  $G$ -module. Corresponding to the extension of  $V_M$ , we also have an extension of  $\Omega^1(M)$ :

$$\begin{aligned} \widetilde{\Omega^1(M)} &= (G \otimes \widetilde{V}_M)^H = (G \otimes (V_M \oplus X))^H \\ &= (G \otimes V_M)^H \oplus (G \otimes X)^H = \Omega^1(M) \oplus (G \otimes X). \end{aligned}$$

From the definition of the module action in (16), we get that, for  $g \in G$ ,

$$\begin{aligned} (1 \otimes X)g &= g_{(1)} \otimes \varepsilon(g_{(2)})X - g_{(1)} \otimes \varepsilon(g_{(2)}) + g_{(1)} \otimes g_{(2)} \\ &= g \otimes X + g_{(1)} \otimes g_{(2)} - g \otimes 1 = g \otimes X + dg. \end{aligned} \quad (17)$$

(From now on, by abuse of notation, we will write  $X$  for  $1 \otimes X$ .) From (17), it is easy to see that

$$dg = Xg - gX. \quad (18)$$

Now since  $\widetilde{V}_M$  is a crossed module, we have a braiding for it extending  $\Psi$ . Moreover, Lemma 6.2 shows that it also induces a braiding for  $\widetilde{\Omega^1(M)}$ . The corresponding exterior algebra clearly contains  $\Omega^\bullet(M)$ . Moreover, it is easy to see that  $\Psi(X \otimes X) = X \otimes X$ . This gives us that  $A_2(X \otimes X) = -X \otimes X + X \otimes X = 0$ , and so

$$X \wedge X = 0. \quad (19)$$

Let  $\omega$  be an element of the exterior algebra extending  $\widetilde{\Omega^1(M)}$ . We set  $d\omega = [X, \omega]_{\text{grad}}$ , where  $[\cdot, \cdot]_{\text{grad}}$  is the graded commutator

$$[X, \omega]_{\text{grad}} = \begin{cases} X \wedge \omega - \omega \wedge X, & \text{for } \omega \text{ a form of even degree;} \\ X \wedge \omega + \omega \wedge X, & \text{for } \omega \text{ a form of odd degree.} \end{cases}$$

It is clear from (18) that  $d$  extends the first order exterior derivative of  $\Omega^1(M)$ . Moreover, for  $\omega$  an even form, (19) implies that

$$\begin{aligned} d(d\omega) &= d(X \wedge \omega - \omega \wedge X) = X \wedge X \wedge \omega + X \wedge \omega \wedge X - X \wedge \omega \wedge X - \omega \wedge X \wedge X \\ &= X \wedge X \wedge \omega - \omega \wedge X \wedge X = 0. \end{aligned}$$

An exactly analogous calculation will establish that  $d^2\omega = 0$ , for  $\omega$  an odd form. It is routine to verify that the other conditions for a differential calculus hold.

To end the proof we have to show that  $d\omega \in \Omega^\bullet(M)$ , for all  $\omega \in \Omega^\bullet(M)$ . Clearly, we only need to verify this for the special case of  $\omega = g_0 dg_1 \wedge \cdots \wedge dg_k \in \Omega^k(M)$ , for  $g_0, \dots, g_k \in \Omega^k(M)$ ,  $1 \leq k \leq N$ . But this is obvious since

$$d(g_0 dg_1 \wedge \cdots \wedge dg_k) = dg_0 \wedge dg_1 \wedge \cdots \wedge dg_k$$

Moreover, this also establishes the uniqueness of  $d$ .  $\square$

#### 6.4 The Exterior Algebra of $\Omega_q^1[U_{N-1}]$

We will now show that  $\Omega_q^1(U_N)$  admits a Woronowicz braiding, and then use it to construct a total calculus.

**Lemma 6.4** *The calculus  $\Omega_q^1(U_N)$  admits a Woronowicz braiding  $\Psi$ . In the corresponding exterior algebra, the subalgebra of relations between the left-invariant one-forms is the bi-ideal generated by the elements*

$$e_{ij} \wedge e_{kl} + e_{kl} \wedge e_{ij}, \quad i, j, k, l = 1, \dots, N. \quad (20)$$

**Proof.** We need to show that  $\text{Ad}_R$  maps each of the generators of  $I_{U_N}$  given in (8) into  $I_G \otimes G + \Lambda_G^1 \otimes \ker(\triangleleft)$ . For  $i \neq j, k \neq l$ , we have

$$\text{Ad}_R(u_j^i u_l^k) = \sum_{a,b,c,d=1}^N u_b^a u_d^c \otimes S(u_a^i u_c^k) u_j^b u_l^d = \sum_{a,b,c,d=1}^N u_b^a u_d^c \otimes S(u_c^k) S(u_a^i) u_j^b u_l^d.$$

When  $i = a, k = c, b = j, d = l$ , we get  $u_j^i u_l^k \otimes S(u_a^i u_c^k) u_j^j u_l^l$ , which is contained in  $I_G \otimes G$ . Otherwise,  $u_b^a u_d^c \otimes S(u_c^k) S(u_a^i) u_j^b u_l^d$  is contained in  $\ker(\triangleleft)$ . Thus,  $\text{Ad}_R(u_j^i u_l^k)$  is contained in  $I_G \otimes G + \Lambda_G^1 \otimes \ker(\triangleleft)$ . That the same holds for the other generators is established similarly, showing that the calculus admits a Woronowicz braiding.

The action of  $\Psi$  on the element  $e_{ij} \otimes e_{kl}$ , for  $i \neq j, k \neq l$  is given by

$$\begin{aligned} \Psi(e_{ij} \otimes e_{kl}) &= \Psi(\overline{u_j^i} \otimes \overline{u_l^k}) = \sum_{a,b=1}^N \overline{u_b^a} \otimes \overline{u_j^i} \triangleleft (S(u_a^k) u_l^b) \\ &= \overline{u_l^k} \otimes \overline{u_j^i} \triangleleft (S(u_k^k) u_l^l) = \overline{u_l^k} \otimes \overline{u_j^i} = e_{kl} \otimes e_{ij}. \end{aligned}$$

For  $e_{ij} \otimes e_{kk}$ , the braiding acts as

$$\begin{aligned}\Psi(e_{ij} \otimes e_{kk}) &= \Psi(\overline{u_j^i} \otimes \overline{u_k^k} - 1) = \sum_{a,b=1}^N \overline{u_b^a} \otimes \overline{u_j^i} \triangleleft (S(u_a^k)u_k^b) - 1 \otimes \overline{u_j^i} \\ &= \overline{u_k^k} \otimes \overline{u_j^i} \triangleleft (S(u_k^k)u_k^k) - 1 \otimes \overline{u_j^i} = \overline{u_k^k} \otimes \overline{u_j^i} - 1 \otimes \overline{u_j^i} \\ &= \overline{u_k^k - 1} \otimes \overline{u_j^i} = e_{kk} \otimes e_{ij}.\end{aligned}$$

Similar calculations will show that

$$\Psi(e_{ii} \otimes e_{kl}) = e_{kl} \otimes e_{ii}, \quad \Psi(e_{ii} \otimes e_{kk}) = e_{kk} \otimes e_{ii}.$$

The relations in (20) now follow directly.  $\square$

From this it is clear that a basis of  $\Lambda_{U_N}^k$  is given by the set:

$$\{e_{(i_1, j_1)} \wedge \cdots \wedge e_{(i_k, j_k)} \mid (i_1, j_1) < (i_2, j_2) < \cdots < (i_k, j_k)\},$$

and so, the exterior algebra has classical dimension.

As a quick test we look at the module  $\Omega^{N^2}[U_N]$  of top forms: It is easy to see that the framing shows it is isomorphic to  $\mathbf{C}_q[U_N]$ , as it should be.

## 7 Complex Structures

We now introduce the notion of an almost-complex structure for a calculus over a  $*$ -algebra. As one would expect, this generalises the classical notion of an almost-complex structure for a manifold.

**Definition 7.1.** An *almost-complex structure* for a  $*$ -differential calculus  $\Omega^\bullet(A)$  over a  $*$ -algebra  $A$ , is a  $\mathbf{Z}^2$ -algebra grading  $\bigoplus_{(p,q) \in \mathbf{Z}^2} \Omega^{(p,q)}(A)$  for  $\Omega^\bullet(A)$  such that

1.  $\Omega^k(A) = \bigoplus_{p+q=k} \Omega^{(p,q)}(A)$ ,
2.  $*(\Omega^{(p,q)}(A)) \subseteq \Omega^{(q,p)}(A)$ .

We call the elements of  $\Omega^{(p,q)}(A)$  the  $(p, q)$ -forms.

One would expect that there is a natural way for the exterior derivative to interact with the complex structure. For this we follow [1], and generalise the classical notion of integrability for an almost-complex structure.

**Lemma 7.2** *For an almost-complex structure  $\bigoplus_{(p,q) \in \mathbf{Z}^2} \Omega^{(p,q)}(A)$ , the two conditions*

1.  $d(\Omega^{(1,0)}(A)) \subseteq \Omega^{(2,0)}(A) \oplus \Omega^{(1,1)}(A)$ ,

$$2. \, d(\Omega^{(0,1)}(A)) \subseteq \Omega^{(1,1)}(A) \oplus \Omega^{(0,2)}(A),$$

are equivalent. If these conditions hold for an almost-complex structure, then we say that it is integrable. We will usually call an integrable almost-complex structure a complex structure.

**Proof.** Let us prove that 1 implies 2: For  $\omega \in \Omega^{(0,1)}(A)$ , we have  $d\omega = (d\omega^*)^*$ . Now  $\omega^* \in \Omega^{(1,0)}(A)$ , and since we are assuming 1, this implies that  $d\omega^* \in \Omega^{(2,0)}(A) \oplus \Omega^{(1,1)}(A)$ . Consequently,  $(d\omega^*)^* \in \Omega^{(1,1)}(A) \oplus \Omega^{(0,2)}(A)$ , and so, 2 follows. The proof in other direction is exactly analagous.  $\square$

The following lemma shows some of the consequences of requiring integrability. Moreover, it shows how our definition of complex structure relates to the one found in [25, 26].

**Lemma 7.3** *If  $\bigoplus_{(p,q) \in \mathbb{Z}^2} \Omega^{(p,q)}(A)$  is a complex structure for a calculus  $\Omega^\bullet(A)$ , then  $\bigoplus_{(p,q) \in \mathbb{Z}^2} \Omega^{(p,q)}(A)$  has a double complex structure  $(\bar{\partial}, \partial)$ , defined by*

$$\bar{\partial}(\omega) = \text{proj}_{(p+1,q)}(d(\omega)), \quad \partial(\omega) = \text{proj}_{(p,q+1)}(d(\omega)), \quad \omega \in \Omega^{(p,q)}(M),$$

where  $\text{proj}_{(a,b)} : \Omega^\bullet(M) \rightarrow \Omega^{(a,b)}(M)$  is the projection onto  $\Omega^{(a,b)}(M)$ . Moreover,  $d = \partial + \bar{\partial}$ , and both  $\partial$  and  $\bar{\partial}$  satisfy the graded Liebniz rule.

**Proof.** We begin by proving that  $d = \partial + \bar{\partial}$ : From (5), we have that any element of  $\Omega^k(A)$  is a sum of elements of the form

$$f_0 df_1 \wedge \cdots \wedge df_k = f_0(\partial f_1 + \bar{\partial} f_1) \wedge \cdots \wedge (\partial f_k + \bar{\partial} f_k), \quad f_i \in A.$$

Since each  $\Omega^{(p,q)}(A)$  is closed under multiplication by the zero form  $f_0$ , every element of  $\Omega^k(A)$  must be spanned by products of  $p$  elements of  $\Omega^{(1,0)}(A)$ , and  $q$  elements of  $\Omega^{(0,1)}(A)$ , such that  $p + q = k$ . The properties of a  $\mathbb{Z}^2$ -algebra grading then imply that each  $\Omega^{(p,q)}(A)$  is spanned by products of  $p$  elements of  $\Omega^{(1,0)}(A)$ , and  $q$  elements of  $\Omega^{(0,1)}(A)$ . From the Liebniz rule, and the assumption of integrability, it now follows that  $d\nu \in \Omega^{(p+1,q)}(A) \oplus \Omega^{(p,q+1)}(A)$ . for any  $\nu \in \Omega^{(p,q)}(A)$ . Thus, we must have that  $d = \bar{\partial} + \partial$ . From  $d = \partial + \bar{\partial}$ , we see that

$$0 = d^2 = (\partial + \bar{\partial}) \circ (\partial + \bar{\partial}) = \partial^2 + (\bar{\partial}\partial + \partial\bar{\partial}) + \bar{\partial}^2.$$

For any  $\omega \in \Omega^k(M)$ , it is easy to see that any non-zero images of  $\omega$  under  $\partial^2$ ,  $\bar{\partial}\partial + \partial\bar{\partial}$ , and  $\bar{\partial}^2$ , would lie in complementary subspaces of  $\Omega^{k+2}(M)$ . Thus, it must hold that

$$\partial^2 = 0, \quad \bar{\partial}\partial = -\partial\bar{\partial}, \quad \bar{\partial}^2 = 0,$$

showing that we have a double complex. That  $\partial$  and  $\bar{\partial}$  satisfy the graded Liebniz rule is established similarly.  $\square$

## 7.1 Framed Complex Structures

We will now introduce a special type of complex structure for calculi over the base of quantum homogeneous bundle. It is a very natural construction, and is based on the complex structure of the  $\mathbf{C}_q[\mathbf{CP}^1]$  presented in [37]. Moreover, the conditions required for the construction to work are easily tested for.

**Theorem 7.4** *Let  $M$  be a quantum framed manifold with a right-absorbing total calculus  $\Omega^\bullet(M)$ . If the cotangent space  $V_M$  of  $M$  can be decomposed into a direct-sum  $V_M = V^{(1,0)} \oplus V^{(0,1)}$ , such that  $V^{(1,0)}$  and  $V^{(0,1)}$  are both right  $G$ -submodules of  $V_M$ ;  $(V_M^{(1,0)})^* = V_M^{(0,1)}$ ; and*

$$\Delta_M(V_M^{(1,0)}) \subseteq V_M^{(1,0)} \otimes H, \quad \Delta_M(V_M^{(0,1)}) \subseteq V_M^{(0,1)} \otimes H, \quad (21)$$

*then  $\Omega^\bullet(M)$  has an almost-complex structure  $\Omega^{(p,q)}(M) = (G \otimes V_M^{(p,q)})^H$ , where, for  $S_k$  the group of permutations on  $k$  objects,*

$$V^{\otimes(p,q)} = \text{span}_{\mathbf{C}}\{\overline{m_1} \otimes \cdots \otimes \overline{m_k} \mid \overline{m_{\pi(1)}} \otimes \cdots \otimes \overline{m_{\pi(k)}} \in (V^{(1,0)})^{\otimes p} \otimes (V^{(0,1)})^{\otimes q}, \pi \in S_k\},$$

*and  $V^{(p,q)}$  is the image of  $V_M^{\otimes(p,q)}$  in  $V_M^k$ . Each  $\Omega^{(p,q)}(M)$  is framed by  $V^{(p,q)}$ , and the restriction of  $\Delta_M \otimes k$  to  $V^{(p,q)}$ . We call such an almost-complex structure a framed almost-complex structure.*

**Proof.** Let us first show that we have a  $\mathbf{Z}^2$ -graded algebra: Since  $V_M^{(1,0)}$  and  $V_M^{(0,1)}$  are closed under the right action of  $M$ , it is clear that  $V_M^{(p,q)}$  is closed under the right action of  $M$ . Thus, for any  $\sum_i f_i \otimes \overline{v_i} \in (G \otimes V_M^{(p,q)})^H$ ,  $m \in M$ , we must have

$$\left(\sum_i f_i \otimes \overline{v_i}\right)m = \sum_i f_i m_{(1)} \otimes \overline{v_i m_{(2)}} \in (G \otimes V_M^{(p,q)})^H.$$

Hence,  $\Omega^{(p,q)}(M)$  must be closed under right multiplication by 0-forms. The question of left multiplication is trivial.

To show that multiplication of higher forms respects the grading, consider the map from  $(G \otimes V^k)^H \times (G \otimes V^l)^H$  to  $(G \otimes V^{k+l})^H$  induced by the multiplication in  $\Omega^\bullet(M)$ :

$$\left(\sum_i f_i \otimes \overline{v_i}, \sum_j g_j \otimes \overline{w_j}\right) \mapsto \sum_{i,j} f_i (g_j)_{(1)} \otimes \overline{v_i (g_j)_{(2)}} \wedge \overline{w_j}.$$

For any  $\overline{v_i} \in V_M^{(p,q)}$ , with  $p+q=k$ , right closure gives us that  $\overline{v_i (f_j)_{(2)}} \in V_M^{(p,q)}$ . Thus, for  $\overline{w_j} \in V_M^{(r,s)}$ , with  $r+s=l$ , we have

$$\overline{v_i (f_j)_{(2)}} \wedge \overline{w_j} \in V_M^{(p+m, q+s)}.$$

Hence, for any  $\omega \in \Omega^{(p,q)}(M)$ ,  $\omega' \in \Omega^{(r,s)}(M)$ , we have  $\omega \wedge \omega' \in \Omega^{(p+r, q+s)}(M)$ .

We now turn to verifying that (1) holds: It is clear that  $V_M^{\otimes k} = \bigoplus_{p+q=k} V_M^{\otimes(p,q)}$ . Moreover, since  $V_M^{(p,q)}$  is right closed, (21) implies  $\Psi(V_M^{\otimes(p,q)}) \subseteq V_M^{\otimes(p,q)}$ . Thus, we have that  $V_M^k = \bigoplus_{p+q=1} V_M^{(p,q)}$ . It now follows from (21) that

$$\Omega^k(M) = (G \otimes (\bigoplus_{p+q=1} V_M^{(p,q)}))^H = \bigoplus_{p+q=1} (G \otimes V_M^{(p,q)})^H = \bigoplus_{p+q=1} \Omega^{(p,q)}(M).$$

Finally, we come to the  $*$ -structure: Consider the map

$$*: G \otimes V_M \rightarrow G \otimes V_M, \quad g \otimes \bar{v} \mapsto g^* \otimes \overline{v^*}.$$

It is easy to see that its restriction to  $(G \otimes V_M)^H$  corresponds to the  $*$ -structure on  $\Omega^{(p,q)}(M)$ . Since  $(V_M^{(1,0)})^* = V_M^{(0,1)}$ , it is clear that we must have  $*(\Omega^{(1,0)}(M)) = \Omega^{(0,1)}(M)$  as required.  $\square$

Now it is natural to ask when a framed almost-complex structure is integrable. The following lemma gives us an easily verifiable criterion.

**Corollary 7.5** *A framed almost-complex structure is integrable if, and only if, for any linear complement  $V_C$  to  $V_M^2$  in  $\Lambda_G^2$ ,*

$$\overline{v_{(1)}} \wedge \overline{v_{(2)}} \in V_M^{(2,0)} \oplus V_M^{(1,1)} \oplus V_C, \quad \text{for all } \bar{v} \in V^{(1,0)}, \quad (22)$$

or equivalently

$$\overline{v'_{(1)}} \wedge \overline{v'_{(2)}} \in V_M^{(1,1)} \oplus V_M^{(0,2)} \oplus V_C, \quad \text{for all } \bar{v'} \in V^{(0,1)}, \quad (23)$$

We call an integrable framed almost-complex structure a framed complex structure.

**Proof.** It is clear that the corollary would follow if we could show that  $d(\bar{\partial}m)$  were contained in  $\Omega^{(2,0)}(M) \oplus \Omega^{(1,1)}(M)$ , for all  $m \in M$ . Now let  $\{e_i^-\}_{i=1}^k$  be a basis of  $V^{(1,0)}(M)$ , for which  $\bar{\partial}m = \sum_{i=1}^k g_i e_k^-$ , with  $g_i \in G$ . The Liebniz rule gives us that

$$d(\bar{\partial}m) = d\left(\sum_{i=1}^k g_i e_k^-\right) = \sum_{i=1}^k (dg_i \wedge e_k^- + g_i de_k^-).$$

While  $d(\bar{\partial}m)$  is of course contained in  $\Omega^\bullet(M)$ , the same is not in general true for  $\sum_{i=1}^k dg_i \wedge e_k^-$ , and  $\sum_{i=1}^k g_i \wedge de_k^-$ . However, the corollary would still follow if we could show that  $\sum_{i=1}^k dg_i \wedge e_k^-$ , and  $\sum_{i=1}^k g_i \wedge de_k^-$ , were contained in  $\Omega_{V_M^{(2,0)}} \oplus \Omega_{V_M^{(1,1)}} \oplus \Omega_{V_C}$ , where

$$\Omega_{V_M^{(2,0)}} = G \otimes V_M^{(2,0)}, \quad \Omega_{V_M^{(1,1)}} = G \otimes V_M^{(1,1)}, \quad \Omega_{V_C} = G \otimes V_C.$$

As a little thought will confirm,  $\sum_{i=1}^k dg_i \wedge e_k^-$  is indeed contained in  $\Omega_{V_M^{(2,0)}} \oplus \Omega_{V_M^{(1,1)}} \oplus \Omega_{V_C}$ . The Maurer–Cartan formula now tells us that if (22) holds, then the same is true for  $\sum_{i=1}^k g_i de_k^-$ .

The derivation of the equivalent condition in (23) is exactly analogous.  $\square$



## 7.2 A Framed Complex Structure for the Quantum Projective Spaces

Let us denote

$$V_{\mathbf{C}P^{N-1}}^{(1,0)} = \mathbf{C}\{e_{i1} \mid i = 1, \dots, N\}, \quad V_{\mathbf{C}P^{N-1}}^{(0,1)} = \mathbf{C}\{e_{1i} \mid i = 1, \dots, N\}.$$

Routine calculation will show that

$$\Delta_{\mathbf{C}P^{N-1}}(V_{\mathbf{C}P^{N-1}}^{(1,0)}) \subseteq V_{\mathbf{C}P^{N-1}}^{(1,0)} \otimes \mathbf{C}_q[U_r] \otimes \mathbf{C}_q[U_{N-r}],$$

and

$$\Delta_{\mathbf{C}P^{N-1}}(V_{\mathbf{C}P^{N-1}}^{(0,1)}) \subseteq V_{\mathbf{C}P^{N-1}}^{(0,1)} \otimes \mathbf{C}_q[U_r] \otimes \mathbf{C}_q[U_{N-r}].$$

Moreover, since

$$(e_i^+)^* = \overline{u_1^i}^* = \overline{(u_1^i)^*} = \overline{S(u_1^i)} = e_i^-,$$

it is clear that  $(V_{\mathbf{C}P^{N-1}}^{(1,0)})^* = V_{\mathbf{C}P^{N-1}}^{(0,1)}$ . Thus,  $\Omega^\bullet(\mathbf{C}P^{N-1})$  has a framed almost-complex structure.

Let us now show that this complex structure is also integrable: Let  $V_C$  be the complement to  $V_{\mathbf{C}P^{N-1}}$  spanned by the elements  $e_{11}$ , and  $e_{ij}$ , for  $i, j \geq 2$ . Moreover, let  $\text{proj}_{(2,0)}$  be the corresponding projection onto  $V^{(2,0)}$ . It holds that

$$\text{proj}_{(2,0)}(\text{de}_i^+) = \text{proj}_{(2,0)}\left(\sum \overline{u_a^i} \wedge \overline{u_1^a}\right) = 0,$$

and so, condition (22) must hold. Alternatively, one can show that (23) holds using an analagous calculation.

## 8 The Noncommutative Kähler Geometry of $\mathbf{C}_q[\mathbf{C}P^1]$

At this point one can introduce a general notion of Hermitian metric, braided metric contraction, volume form, and Hodge  $*$ -operator (defined as contraction with the volume form). For the specific case of the quantum projective spaces, there is a natural metric generalising the Fubini–Study metric. Moreover, one can give a direct presentation of the associated Hodge  $*$ -map, fundamental form, Lefschetz operator, and co-Lefschetz operator in terms of the basis given above. For the first few examples of the quantum projective spaces, it is possible to verify directly that the Kähler identities hold. For the higher orders, however, a more systematic approach is required. Such an approach is being developed, and will appear in [41]. For now, we present these constructions for the special case of the quantum projective line  $\mathbf{C}_q[\mathbf{C}P^1]$ .

Let  $V_{\mathbf{R}}$  be the real vector space with basis  $\{b_1, b_2\}$ . We define a complex structure  $J$  on  $V_{\mathbf{R}}$  by

$$J(b_1) = b_2, \quad J(b_2) = -b_1.$$

We denote its complexification by  $V$ . As usual, we extend  $J$  by  $\mathbf{C}$ -linearity to a mapping on  $V$ . Its eigenvalues are  $i$  and  $-i$ , with respective eigenspaces

$$V^{(1,0)} = \{v - iJ(v) | v \in V\}, \quad V^{(0,1)} = \{v + iJ(v) | v \in V\}.$$

It is easy to see that  $V^{(1,0)}$  is spanned over  $\mathbf{C}$  by the element  $e^+ = b_1 - ib_2$ , and that  $V^{(0,1)}$  is spanned over  $\mathbf{C}$  by the element  $e^- = b_1 + ib_2$ . In terms of this new basis we have that  $b_1 = \frac{1}{2}(e^- + e^+)$  and  $b_2 = \frac{-i}{2}(e^- - e^+)$ . Thus, we have recovered  $V_{\mathbf{C}P^1}$ , and its complex structure, as the complexification of a real vector space with an almost complex structure.

A metric  $g$  on  $V_{\mathbf{R}}$  is Hermitian with respect to  $J$  if, and only if,  $g(b_i, b_j) = c\delta_{ij}$ , for some non-zero real number  $c$ . We will choose the metric that gives  $\{b_1, b_2\}$  as an orthonormal basis, which is to say, we choose  $c = 1$ . The natural extension of  $g$  to a metric on the exterior algebra of  $V_{\mathbf{R}}$  gives  $e^1 \wedge e^2$  as the top form, and induces a Hodge  $*$ -map defined by  $*(b_1) = b_2$ , and  $*(b_2) = -b_1$ . As usual, we extend this to a  $*$ -map on all  $V_{\mathbf{C}P^1}$  by  $\mathbf{C}$ -linearity. This gives

$$\begin{aligned} *(e^-) &= *(b_1 + ib_2) = b_2 - ib_1 = -i(b_1 + ib_2) = -ie^- \\ *(e^+) &= *(b_1 - ib_2) = b_2 + ib_1 = i(b_1 - ib_2) = ie^+. \end{aligned}$$

We then extend this to a map on  $\Omega^0(\mathbf{C}P^1) \oplus \Omega^1(\mathbf{C}P^1) \oplus \Omega^2(\mathbf{C}P^1)$  as  $\text{id} \otimes *$ . By abuse of notation, we will also denote this map by  $*$ . We see that  $*$  squares to give  $-1$  on  $\Omega^1(\mathbf{C}P^{N-1})$  as it should. We can also extend  $g$  to a metric on  $\Omega_q^1[\mathbf{C}P^1]$  by setting

$$g : \Omega_q^1[\mathbf{C}P^1] \times \Omega_q^1[\mathbf{C}P^1] \rightarrow \mathbf{C}_q[\mathbf{C}P^1], \quad (f \otimes e^\pm, h \otimes e^\pm) \mapsto h^* f g(e^\pm, e^\pm).$$

Note that since  $\Omega^{(1,0)}(\mathbf{C}_q[\mathbf{C}P^1]) \simeq \mathcal{E}_{-1}$ , and  $\Omega^{(1,0)}(\mathbf{C}_q[\mathbf{C}P^1]) \simeq \mathcal{E}_1$ , this is a well-defined map.

Following the classical picture we take the closed two form  $\kappa = -e^+ \wedge e^-$  as the Kähler form. We then define the operator

$$L : \mathbf{C}_q[\mathbf{C}P^1] \rightarrow \Omega^2(\mathbf{C}P^1), \quad f \mapsto f\kappa,$$

and the operator

$$\Lambda : \Omega^2(\mathbf{C}P^1) \rightarrow \mathbf{C}_q[\mathbf{C}P^1], \quad \omega \mapsto - * \circ L \circ *(\omega).$$

We extend  $L$  and  $\Lambda$  to operators on the total calculus by defining them to be zero on all other forms. Moreover, we define the codifferentials by

$$d^* = - * d *, \quad \partial^* = - * \bar{\partial} *, \quad \bar{\partial}^* = - * \partial *.$$

Finally, we define the Laplace operators by

$$\Delta = (d + d^*)^2, \quad \Delta_\partial = (\partial + \partial^*)^2, \quad \Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^*)^2.$$

The following proposition shows that these operators satisfy the generalisation of the Kähler identities for the sphere:

**Proposition 8.1** *We have the following relations:*

$$\begin{aligned} [L, \partial^*] &= i\bar{\partial}, & [L, \bar{\partial}^*] &= -i\partial, & [L, \partial] &= 0, & [L, \bar{\partial}] &= 0, \\ [\Lambda, \partial] &= i\bar{\partial}^*, & [\Lambda, \bar{\partial}] &= -i\partial^*, & [\Lambda, \partial^*] &= 0, & [\Lambda, \bar{\partial}^*] &= 0, \end{aligned}$$

**Proof.** The relations

$$[L, \partial] = [L, \bar{\partial}] = [\Lambda, \partial^*] = [\Lambda, \bar{\partial}^*] = 0$$

are direct consequences of the definition of  $L$  and  $\Lambda$ . The remaining relations are easily verified by direct calculation. We show this for the relation  $[L, \partial^*] = i\bar{\partial}$ : First we note that  $[L, \partial^*]$  has a non-zero action only on  $\mathbf{C}_q[\mathbf{CP}^1]$  and  $\Omega^{(1,0)}(\mathbf{CP}^1)$ . For  $f \in \mathbf{C}_q[\mathbf{CP}^1]$ , we have

$$[L, \partial^*]f = (-L \circ * \bar{\partial} * + * \bar{\partial} * \circ L)f = * \bar{\partial} * \circ Lf = * \bar{\partial} * (f\kappa) = * \bar{\partial} f = i\bar{\partial}f.$$

While for  $f\partial h \in \Omega^{(1,0)}(\mathbf{CP}^1)$ , we have

$$\begin{aligned} [L, \partial^*](f\partial h) &= (-L \circ * \bar{\partial} * + * \bar{\partial} * \circ L)f\partial h = -L \circ * \bar{\partial} * f\partial h \\ &= -iL \circ * \bar{\partial}(f\partial h). \end{aligned}$$

Now  $\bar{\partial}(f\partial h) = ke^+ \wedge e^-$ , for some  $k \in \mathbf{C}_q[\mathbf{CP}^1]$ , and so,

$$[L, \partial^*](f\partial h) = -iL \circ *(ke^+ \wedge e^-) - iL(k) = -ik\kappa = ie^+ \wedge e^-.$$

On the other hand

$$i\bar{\partial}(f\partial h) = ike^+ \wedge e^-,$$

which establishes the relation.  $\square$

**Corollary 8.2** *The Laplace operators are related by  $\Delta = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$ .*

**Proof.** This can be established by direct calculation. Alternatively, one can follow the standard classical proof [21]: First note that

$$-i(\bar{\partial}\partial^* + \partial^*\bar{\partial}) = \bar{\partial}[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\bar{\partial} = \bar{\partial}\Lambda\bar{\partial} - \bar{\partial}\Lambda\bar{\partial} = 0,$$

and similarly  $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$ . This gives that

$$\begin{aligned} \Delta &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\ &= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + (\bar{\partial}\partial^* + \partial^*\bar{\partial}) + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) \\ &= \Delta_\partial + \Delta_{\bar{\partial}}. \end{aligned}$$

It remains to show that  $\Delta_\partial = \Delta_{\bar{\partial}}$ , which is an easy consequence of the proposition:

$$\begin{aligned} -i\Delta_\partial &= -i(\partial\partial^* + \partial^*\partial) = \partial[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\partial = \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial \\ &= \partial\Lambda\bar{\partial} + \bar{\partial}\partial\Lambda - \bar{\partial}\Lambda\partial = [\partial, \Lambda]\bar{\partial} + \bar{\partial}[\partial, \Lambda] = -i\bar{\partial}^*\bar{\partial} - i\bar{\partial}\partial^* = -i\Delta_{\bar{\partial}}. \end{aligned}$$

□

Finally, we note that the *irreducible spinor bundle* of  $\mathbf{C}_q[\mathbf{CP}^1]$  is given by

$$(\Omega^{(0,0)}(\mathbf{CP}^1) \oplus \Omega^{(0,1)}(\mathbf{CP}^1)) \otimes \mathcal{E}_{-1} \simeq \mathcal{E}_{-1} \otimes \mathcal{E}_1.$$

While the *spin Dirac operator* is given by a twist of the Dirac–Dolbeault  $\bar{\partial} + \bar{\partial}^*$  by the standard monopole connection (see [37]).

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